On Graded rings

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Basic definitions

Definition

Let G be a group with neutral element e. A ring R is said to be graded by G, or graded of type G, if there is a family $\{R_g\}_{g\in G}$ of additive subgroups R_g of R such that

$$R = \bigoplus_{g\in G} R_g$$

and

$$R_g R_h \subseteq R_{gh} \tag{1}$$

for all $g, h \in G$. The additive subgroup R_g is called the homogeneous component of R of degree $g \in G$.

Basic definitions

the rest..

- *The set* $h(R) = \bigcup_{g\in G} R_g$ *is the set of homogeneous elements of* R. *A nonzero element* $x \in R_g$ *is said to be homogeneous of degree* g *and we write* $deg(x) = g$. *Each element* $r \in R$ *has a unique decomposition* $r = \sum_{g\in G} r_g$ *with* $r_g \in R_g$ *for all* $g \in G$, *and the sum is finite, i.e. almost all* r_g *are zero. The support of* r *in* G *is denoted by* $supp(r) := \{g \in G | r_g \neq 0\}$.

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- *The set* $h(R) = \bigcup_{g \in G} R_g$ *is the set of* *homogeneous elements* *of* R. *A nonzero element* $x \in R_g$ *is said to be homogeneous of degree* g *and we write* $deg(x) = g$. *Each element* $r \in R$ *has a unique decomposition* $r = \sum_{g \in G} r_g$ *with* $r_g \in R_g$ *for all* $g \in G$, *and the sum is finite, i.e. almost all* r_g *are zero. The support of* r *in* G *is denoted by* $supp(r) := \{g \in G | r_g \neq 0\}$.
- *If* R *is a graded ring of type* G *such that* $R_h R_g = R_{hg}$ *holds for all* $h, g \in G$, *then we say that* R *is* *strongly graded* *of type* G *or strongly graded of type* G.

Basic definitions

Definition

Let $R = \bigoplus_{g \in G} R_g$ *be a graded ring. A subring* S *of* R *is called a graded subring of* R *if* $S = \sum_{g \in G}(R_g \cap S)$*. Equivalently,* S *is graded if for every element* $f \in S$ *all the homogeneous components of* f *(as an element of* R*) are in* S*.*

Basic definitions

Examples

Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ *be a graded ring and* $f_1, \cdots, f_d$ *homogeneous elements of* R *of degrees* $\beta_1, \cdots, \beta_d$ *respectively. Then* $S = R_0[f_1, \cdots, f_d]$ *is a graded subring of* R*, where* $S_n = \{\sum_{m \in \mathbb{N}^d} r_m f_1^{m_1} \cdots f_d^{m_d} | r_m \in R_0$ *and* $\beta_1 m_1 + \cdots + \beta_d m_d = n\}$.

Basic definitions

Remark

Note that any ring R is strongly graded of type G by chosing the trivial group $G = \{e\}$ as grading group and putting $R_e = R$.

Basic definitions

Proposition

Let $R = \bigoplus_{g \in G} R_g$ *be a graded ring of type* G*. The following assertions hold:*

- R_e *is subring and* $1_R \in R_e$*.*

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Proposition

Let $R = \bigoplus_{g \in G} R_g$ be a graded ring of type G. The following assertions hold:

- *R_e is subring and $1_R \in R_e$.*
- *If $r \in U(R)$ is a homogeneous element of degree $h \in G$, then its inverse r^{-1} is a homogeneous element of degree h^{-1}.*
- *R is a strongly graded ring of type G if and only if $1_R \in R_g R_{g^{-1}}$ for each $g \in G$.*

Basic definitions

Proof

- *It is clear that* R_e *is a subring of* R*. To prove that* $1_R \in R_e$*, let* $1_R = \sum_{g \in G} r_g$ *be the decomposition of* 1_R *with* $r_g \in R_g$ *for each* $g \in G$*. For any* $s_h \in R_h$*,* $h \in G$*, we have*

$$s_h = s_h 1_R = \sum_{g \in G} s_h r_g$$

and $s_h r_g \in R_{hg}$*. Consequently, for each* $g \neq e$ *we have* $s_h r_g = 0$ *and hence* $s r_g = 0$ *for any* $s \in R$*. In particular, for* $s = 1$ *we obtain* $r_g = 0$ *for any* $g \neq e$*. Thus,* $1_R = r_e \in R_e$*.*

Basic definitions

the rest..

- *Assume that* $r_h \in U(R) \cap R_h$ *for some* $h \in G$*. If* $r_h^{-1} = \sum_{g \in G} r_g$ *with* $r_g \in R_g$ *then* $1_r = r_h r_h^{-1} = \sum_{g \in G} r_h r_g$*. Since* $1_R \in R_e$ *and* $r_h r_g \in R_{hg}$ *we have* $r_h r_g = 0$ *for* $g \neq h^{-1}$*. Scine* $r_h \in U(R)$ *we get that* $r_g = 0$ *for* $g \neq h^{-1}$ *and therfore* $r_h^{-1} = r_{h^{-1}} \in R_{h^{-1}}$*.so that shows that* r_h^{-1} *is a homogeneous element of degree* h^{-1}*.*

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- *Assume that* $r_h \in U(R) \cap R_h$ *for some* $h \in G$*. If* $r_h^{-1} = \sum_{g\in G} r_g$ *with* $r_g \in R_g$ *then* $1_r = r_h r_h^{-1} = \sum_{g\in G} r_h r_g$*. Since* $1_R \in R_e$ *and* $r_h r_g \in R_{hg}$ *we have* $r_h r_g = 0$ *for* $g \neq h^{-1}$*. Scine* $r_h \in U(R)$ *we get that* $r_g = 0$ *for* $g \neq h^{-1}$ *and therfore* $r_h^{-1} = r_{h^{-1}} \in R_{h^{-1}}$*.so that shows that* r_h^{-1} *is a homogeneous element of degree* h^{-1}*.*
- *Suppose that* $1_R \in R_g R_{g^{-1}}$ *for each* $g \in G$*. For each* $g, h \in G$ *we have* $R_{gh} = 1_R R_{gh} \subseteq R_g R_{g^{-1}gh} = R_g R_h \subseteq R_{gh}$ *which shows that* $R_{gh} = R_g R_h$*, and hence* R *is strongly graded of type* G*. Conversely, if* R *is strongly graded of type* G*, then it follows from i) that that* $1_R \in R_e = R_g R_{g^{-1}}$ *for each* $g \in G$

Basic definitions

Remark

- *Note that R is an R_0-algebra.*

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- *Note that R is an R_0-algebra.*
- *Note that each homogeneous component R_g has a natural R_0-module structure.*

Basic definitions

Proposition

Let $R = \bigoplus_{g \in G} R_g$ *be a strongly graded ring of type* G*. If* $a \in R$ *is such that*

$$aR_g = \{0\} \text{ or } R_g a = \{0\}.$$

for some $g \in G$*, then* $a = 0$*.*

Basic definitions

Proof

Suppose that $aR_g = \{0\}$ *for some* $g \in G$, $a \in R$. *We then have* $aR_gR_{g^{-1}} = \{0\}$ *or* $aR_e = \{0\}$. *From the fact that* $1_R \in R_e$, *we conclude that* $a = 0$.*The other case is treated analogously.*

Basic definitions

Proposition

Let $R = \bigoplus_{g \in G} R_g$ *be a graded ring of type* G *and* N *be a normal subgroup of* G*, then* R *can be regarded as graded ring of type* G/N*, where the homogeneous components are given by*

$$R_{gN} = \bigoplus_{x \in gN} R_x$$

for $gN \in G/N$*.*

Basic definitions

Proof

Let S be a transversal for N in G. It is obvious that

$$G = \bigoplus_{s \in S} R_{sN}$$

For any $s_1, s_2 \in S$, we have

$$R_{s_1 N} R_{s_2 N} = (\bigoplus_{x \in s_1 N} R_x)(\bigoplus_{y \in s_2 N} R_y) \subseteq \bigoplus_{(x,y) \in s_1 N \times s_2 N} R_{xy} = \bigoplus_{z \in s_1 s_2 N} R_z = R_{s_1}$$

which shows that R is graded ring of type G/N.

Basic definitions

Definition

- *Analogously there is the notion of graded k-associative algebra over any commutative ring k. Specifically for k a field a graded algebra is a monoid in graded vector spaces over k.*

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- *Analogously there is the notion of graded k-associative algebra over any commutative ring k. Specifically for k a field a graded algebra is a monoid in graded vector spaces over k.*
- *A graded Lie algebra is an ordinary Lie algebra g, together with a gradation of vector spaces* $g = \bigoplus_{i \in \mathbb{Z}} g_i$ *such that the Lie bracket respects this gradation* $[g_i, g_j] \subseteq g_{i+j}$.

Examples of graded rings

Examples

- *The ring* $R := k[X]$ *of polynomials with coefficients in* k *is a graded ring of type* $\mathbb{Z}$, *where* $R_n = kX^n$ *if* $n > 0$ *and* $R_n = 0$ *if* $n < 0$. *So* $k[X] = \bigoplus_{n \in \mathbb{Z}} kX^n$.

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- *The polynomial ring* $R := k[T_1, \cdots, T_n]$ *in* n *indeterminates with coefficients in* k *is a graded ring of type* $\mathbb{Z}$ *where* R_n *is the set of homogeneous polynomials of degree* n.

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- *The polynomial ring* $R := k[T_1, \cdots, T_n]$ *in* n *indeterminates with coefficients in* k *is a graded ring of type* $\mathbb{Z}$ *where* R_n *is the set of homogeneous polynomials of degree* n.
- *Any ring* R *may be considered as a graded ring of type* G, *for any group* G, *by putting* $R_e = R$, $R_g = 0$ *for* $g \neq e$ *in* G. *Such a ring is said to be trivially* $G-$*Graded*.

Examples

- *Let R be a graded ring of type G. Let R° be the opposite ring for R i.e, R° has the same underlying additive group as R but multiplication in R° is defined by the rule*

$$x \circ y = yx$$

Putting $(R^\circ)_g = R_{g^{-1}}$ make R° a graded ring of type G.

Examples

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Putting $(R^\circ)_g = R_{g^{-1}}$ make R° a graded ring of type G.

- *The Lie algebra $\mathfrak{sl}_2(\mathbb{R}) := \{M \in \mathfrak{gl}_2(\mathbb{R}) / tr(M) = 0\}$ is graded by the generators:*

$$X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These satisfy the relations $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$. Hence with $\mathfrak{g}_{-1} := span(X)$, $\mathfrak{g}_0 := span(H)$ and $\mathfrak{g}_1 := span(Y)$, the decomposition $\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ presents $\mathfrak{sl}_2(\mathbb{R})$ as a graded Lie algebra.

Examples of graded rings

- The field $\mathbb{C}$ of the complex numbers can be graded by the group $\mathbb{Z}_2 = \{0, 1\}$:

$$\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}, \mathbb{C}_0 = \mathbb{R}, \mathbb{C}_1 = i\mathbb{R}.$$

The graduation is strong in both cases.

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- The division ring $\mathbb{H}$ of the quaternions can be graded by the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$:

$$\mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$$

$$\mathbb{H}_{(0,0)} = \mathbb{R},\ \ \mathbb{H}_{(0,1)} = i\mathbb{R},\ \ \mathbb{H}_{(1,0)} = j\mathbb{R},\ \ \mathbb{H}_{(1,1)} = k\mathbb{R}.$$

The graduation is strong in both cases.

Basic definitions

the rest..

- *Let* A *be a ring and* $R := M_3(A)$ *the matrix ring over* A*. By putting*

$$R_0 = \begin{pmatrix} A & A & O \\ A & A & 0 \\ 0 & 0 & A \end{pmatrix} \text{ and } R_1 = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & A \\ A & A & 0 \end{pmatrix}$$

one may verify that this defines a strongly gradation of type $\mathbb{Z}_2$ *on* R*.*

Graded Homomorphisms and isomorphisms

Definition

Let $R = \bigoplus_{g\in G} R_g$ and $S = \bigoplus_{h\in G} S_g$ be two graded rings of type G. A morphism of graded rings is a ring homomorphism $\psi : R \longrightarrow S$ such that for all $g \in G$, $\psi(R_g) \subseteq S_g$.

Note that if $f : R \longrightarrow S$ and $g : T \longrightarrow T$ are two graded homomorphisms. It easy to show that $g \circ f : R \longrightarrow T$ is a graded homomorphism.
We can form the category Gr-Rings of graded rings whose objects are graded rings and whose morphisms are graded homomorphisms.

Homogeneous Ideals and Quotient Rings

Definition

Let R be graded ring of type G and I be a left (resp. right) ideal of R. We say that I is graded left (resp. right) ideal of R if $I = \bigoplus_{g \in G} I_n$ where $I_g := I \cap R_g$, for each $g \in G$

Not every right ideal of graded ring must be graded. Consider the positively graded $\mathbb{R}[X]$, with grading $\{X^n R\}_{n>0}$. Since $(1+X) \in (1+X)\mathbb{R}[X]$ and $1 \notin (1+X)\mathbb{R}[X]$. So $1+X$ cannot be written as a sum of homogeneous elements of $(1+X)\mathbb{R}[X]$. thus $(1+x)\mathbb{R}[X]$ is not graded ideal.

Homogeneous Ideals

Proposition

Let R be a graded ring of type G and I be an ideal of R.

- *I is homogeneous if and only if I generated by homogeneous elements.*

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Let R be a graded ring of type G and I be an ideal of R.

- *I is homogeneous if and only if I generated by homogeneous elements.*
- *I is homogeneous elements if and only if for all $f \in I$, also all homogeneous components $f_g \in I$.*

Homogeneous Ideals

Definition

Let R be a graded ring of type G. A graded left (right, two-sided) ideal M of R is a graded-maximal left (right, two-sided) ideal if $M \neq R$ and M is not contained in any other proper graded left (right, two-sided) ideals of R.

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Graded Analogs of Classical Notions

If we replace all elements with merely homogeneous ones and all ideals (submodules) with merely graded ones in definitions of classical notions, then we will obtain their standard graded analogs. These graded analogs are denoted by the prefix gr-.

Gr-Regularity

Recall that, A regular ring in the sense of commutative algebra is a commutative unit ring such that all its localizations at prime ideals are regular local rings.
In contrast, a Von Neumann regular ring is an object of noncommutative ring theory defined as a ring R such that for all a in R, there exists b in R satisfying $a = aba$. Von Neumann regular rings are unrelated to regular rings (or regular local rings) in the sense of commutative algebra.
For example, a polynomial ring over a field is always regular in the sense of commutative algebra, but is certainly not regular in the sense of Von Neumann, since if X is an indeterminate, then the required property is evidently not fulfilled.

Gr-Regularity

Definition

A graded ring R is called gr-regular if $a \in aRa$ for all $a \in h(R)$, i.e., the equation $a = axa$ is solvable about $x \in R$ for all $a \in h(R)$.

- It is clear that if $a \in R_g$ and $a = axa$, then we can replace x by its homogeneous component x_h of $degree(h) = g^{-1}$ and the elements $ax_{g^{-1}}, x_{g^{-1}}a \in R_e$ are homogeneous idempotents of the ring R

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- It is clear that if $a \in R_g$ and $a = axa$, then we can replace x by its homogeneous component x_h of $degree(h) = g^{-1}$ and the elements $ax_{g^{-1}}, x_{g^{-1}}a \in R_e$ are homogeneous idempotents of the ring R
- It is clear that any graded regular ring is gr-regular. At the same time, a gr-regular ring need not be regular.

Gr-Regularity

Theorem

A group ring $R[G]$ is regular if and only if the ring R is regular, the group G is local finite (i.e., each finitely generated subgroup of G is finite), and the order of each subgroup of G is invertible in the ring R.

The group ring $k[\mathbb{Z}]$, where k is a field, is a $\mathbb{Z}$-graded division ring and hence a gr-regular ring, but it is not regular.

Gr-Regularity

Definition

A graded ring R of type G is called

- *right (left) g-faithful ($g \in G$) if for any $r \in h(R) \setminus \{0\}$ there exists $r' \in h(R)$ such that $rr' \in R_g \setminus \{0\}$.*

Gr-Regularity

Definition

A graded ring R *of type* G *is called*

- *right (left) g-faithful* ($g \in G$) *if for any* $r \in h(R) \setminus \{0\}$ *there exists* $r' \in h(R)$ *such that* $rr' \in R_g \setminus \{0\}$.
- *right (left) faithful if* R *is right (left) g-faithful for all* $g \in G$.

Gr-Regularity

Definition

A graded ring R of type G is called

- *right (left) g-faithful ($g \in G$) if for any $r \in h(R) \setminus \{0\}$ there exists $r^{'} \in h(R)$ such that $rr^{'} \in R_g \setminus \{0\}$.*
- *right (left) faithful if R is right (left) g-faithful for all $g \in G$.*
- *g-faithful (faithful) if R is right and left g-faithful (faithful)*

Gr-Regularity

Theorem

Let R be a graded ring of type G. If R is a strongly graded ring and the ring R_e is regular, then the ring R is gr-regular.

Gr-Primary and gr-Semiprimary.

Definition

- *A graded ring R is called gr-prime if the following equivalent conditions hold:*
    i) $IJ \neq 0$ *for all nonzero graded left (right or two-sided) ideals I and J of R.*
    ii) *for all $a, b \in h(R)$, if $aRb = 0$, then $a = 0$ or $b = 0$.*

It is clear that any prime (semiprime) graded ring is gr-prime (gr-semiprime). The converse statement is false.

Gr-Primary and gr-Semiprimary.

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  - i) $IJ \neq 0$ *for all nonzero graded left (right or two-sided) ideals I and J of R.*
  - ii) *for all $a, b \in h(R)$, if $aRb = 0$, then $a = 0$ or $b = 0$.*
- *A graded ring R of type G is called gr-semiprime if the following equivalent conditions hold:*
  - i) $I^2 \neq 0$ *for all nonzero left (right or two-sided) graded ideals I of R.*
  - ii) *for all $a \in h(R)$, if $aRa = 0$, then $a = 0$.*

It is clear that any prime (semiprime) graded ring is gr-prime (gr-semiprime). The converse statement is false.

Gr-Primary and gr-Semiprimary

Theorem

Let R be a ring and G be a group. Then

- *The ring $R[G]$ is prime if and only if the ring R is prime and $\{e\}$ is unique finite normal subgroup of the group G (see [16, Theorem 21], [41, p. 258]).*

A graded field $k[\mathbb{Z}_2]$ is not prime for any field k and is not semiprime if $char(k) = 2$.

Gr-Primary and gr-Semiprimary

Theorem

Let R be a ring and G be a group. Then

- *The ring $R[G]$ is prime if and only if the ring R is prime and $\{e\}$ is unique finite normal subgroup of the group G (see [16, Theorem 21], [41, p. 258]).*
- *The ring $TR[G]$ is semiprime if and only if the ring R is semiprime and the orders of normal subgroups of G are not zero divisors in R. see [41, p. 255], [59, Theorem 13.2]).*

A graded field $k[\mathbb{Z}_2]$ is not prime for any field k and is not semiprime if $char(k) = 2$.

Gr-Primary and gr-Semiprimary

Theorem

Let R be a gr-semiprime ring with finite support. Then

- *R is e-faithful.*

Gr-Primary and gr-Semiprimary

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Let R be a gr-semiprime ring with finite support. Then

- *R is e-faithful.*
- *R_e is semiprime.*

Gr-Primary and gr-Semiprimary

Theorem

Let R be a gr-semiprime ring with finite support. Then

- *R is e-faithful.*
- *R_e is semiprime.*
- *$g \in Supp(R)$ if and only if $g^{-1} \in Supp(R)$ for all $g \in G$.*

Gr-Primary and gr-Semiprimary

Theorem

Let R be graded ring of type G. If G is an ordered group, then R is prime (semiprime) if and only if R is gr-prime (gr-semiprime).

Let R be graded ring of type G and R is an e-faithful (right and left) ring. If the ring R_e is prime (semiprime), then R is gr-prime (gr-semiprime). In fact, if I and J are nonzero graded ideals in R with $IJ \neq 0$, then the e-faithfulness implies that I_e and J_e are nonzero ideals in R_e and $I_eJ_e \neq 0$.

Definition

Let R be graded ring of type G.

- *R is called a graded domain if it does not contain homogeneous zero divisors.*

Each graded domain is gr-prime, and each gr-reduced ring is gr-semiprime. The converse statements are true in the case of commutative rings.

Definition

Let R be graded ring of type G.

- *R is called a graded domain if it does not contain homogeneous zero divisors.*
- *R is called gr-reduced if it does not contain homogeneous nilpotent element.*

Each graded domain is gr-prime, and each gr-reduced ring is gr-semiprime. The converse statements are true in the case of commutative rings.