NOTE ON SHEAF THEORY

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Abstract. In this note, we review some fundamental concepts and constructions in sheaf theory.

1. Presheaves

Notation. *Let X be a topological space. We denote by $\mathcal{T}_X$ the category having for objects the open subsets of X and for morphisms identity maps and inclusions.*
$\mathcal{C}$ will denote a category, which can be the category of sets (also denoted by $\mathcal{S}et$), that of groups (also denoted by $\mathcal{G}p$), that of R-modules (also denoted by $\mathcal{R}ing$), that of R-modules (also denoted by R-$\mathcal{A}lg$), for some ring R.

Definition 1.1. *Let X be a topological space. A presheaf $\mathcal{F}$ on X consists of the following deta :*

i) For every open subset U of X, a set $\mathcal{F}(U)$.
ii) Whenever $U \subseteq V$ are two open subsets of X, a map

$$res_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$

called the restriction map, which satisfies the following conditions :
a) $res_{U,U} = id_{\mathcal{F}(U)}$.
b) Having three open subsets $U \subseteq V \subseteq W$ of X, then $res_{V,U} \circ res_{W,V} = res_{W,U}$

Remarks 1.1. *1) We will mostly write $s_{|U}$ for s when $s \in \mathcal{F}(U)$. The elements of $\mathcal{F}(U)$ are usually called sections of (the presheaf $\mathcal{F}$) over U.*
2) By considering $\mathcal{F}(U)$ as object in some category $\mathcal{C}$ and assuming that $res_{V,U}$ is a morphism between the objects $\mathcal{F}(V)$ and $\mathcal{F}(U)$, we may define more generally a presheaf $\mathcal{F}$ on X into $\mathcal{C}$.
3) Note that we can state definition 1.1 in another way: Let X be a topological space. A presheaf $\mathcal{F}$ on X (into a category $\mathcal{C}$) is a contravariant functor from $\mathcal{T}_X$ into $\mathcal{C}$.

$$\begin{array}{rccc} \mathcal{F}: & \mathcal{T}_X & \longrightarrow & \mathcal{C} \\ & U & \longmapsto & \mathcal{F}(U) \end{array}$$

Examples 1.1. *1) For a topological space, a presheaf $\mathcal{C}_X$ of $\mathbb{R}$-algebras on X is defined by assigning to every open $U \subseteq X$ the set of continuous functions $U \longrightarrow \mathbb{R}$.*

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2) *Let X be a variety, we previously considered the presheaf of k-algebras $\mathcal{O}_X$. For any open $U \subseteq X$, $\mathcal{O}_X(U)$ is the k-algebra of regular functions. If X be an affine variety we have $\mathcal{O}_X(U) = k[U]$.*
3) *Let X be a topological space, the formula :*

$$U \longmapsto \begin{cases} \mathbb{Z} & \text{if} \quad U = X \\ \{0\} & \text{otherwise} \end{cases}$$

defines a presheaf of abelian groups on X.

Although it is possible to define a presheaf of a topological space X into an arbitrary category $\mathcal{C}$, we will be interested in what follows only in cases where the objects of $\mathcal{C}$ are sets (that could have an additional structure) and the morphisms $res_{V,U}$ are maps (which are morphisms for the extra structure on $\mathcal{F}(V)$ and $\mathcal{F}(U)$.

Definition 1.2. *Let $\mathcal{F}$ be a presheaf on X, a subpresheaf $\mathcal{G}$ (of $\mathcal{F}$) is a presheaf on X such that $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ for every open $U \subseteq X$, and such that the restriction maps of $\mathcal{G}$ are induced by those of $\mathcal{F}$.*

Example 1.1. *If U is an open subset of X, every presheaf $\mathcal{F}$ on X induces, in an obvious way, a presheaf $\mathcal{F}_U$ on U by setting $\mathcal{F}_{|U}(V) = \mathcal{F}(V)$ for every open subset V of U. This is the restriction of $\mathcal{F}$ to U.*

1.1. **Morphisms of presheaves.**

Definition 1.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X. A morphism of presheaves ψ from $\mathcal{F}$ to $\mathcal{G}$ consists of the datum, for all open U of X, of a morphism $\psi(U)$ from $\mathcal{F}(U)$ to $\mathcal{G}(U)$, so that the diagram*

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\psi(V)"] \arrow[d, "res_{V,U}"'] & \mathcal{G}(V) \arrow[d, "res_{V,U}"] \\ \mathcal{F}(U) \arrow[r, "\psi(U)"'] & \mathcal{G}(U) \end{tikzcd}$$

commutes for any pair (U,V) of open subsets of X such that $U \subseteq V$.

Remarks 1.2. *1) The commutativity of the diagram is written : $\psi(V)(s)_{|U} = \psi(U)(S_{|U})$, where $s \in \mathcal{F}(V)$.*

2) *Morphisms of presheaves can be composed. So that presheaves on the topological space X form a category, that we will denote by $\mathcal{P}reSh_X$.*
3) *A morphism $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ between two presheaves $\mathcal{F}$ and $\mathcal{G}$ is an isomorphism if it has a two-sided inverse i.e, a morphism $\phi : \mathcal{G} \longrightarrow \mathcal{F}$ such that $\psi \circ \phi = id_{\mathcal{G}}$ and $\phi \circ \psi = id_{\mathcal{F}}$.*

Definition 1.4. *Assume $\mathcal{C}$ has direct limits. The stalk of a presheaf $\mathcal{F}$ at a point $x \in X$ is*

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

The direct limit is taken over open neighborhoods of x, and restriction maps between them. Given a section $s \in \mathcal{F}(U)$, and a point $x \in U$, we let $s_x \in \mathcal{F}_x$ denote the image of s under the natural morphism

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}_x \\ s & \longmapsto & s_x \end{array}$$

An element of the stalk is called a germ.
More generally, if $Y \subseteq X$ is a closed and irreducible subset. Then, we set

$$\mathcal{F}_Y := \varinjlim_{U \cap Y \neq \emptyset} \mathcal{F}(U)$$

Notation. *Let X be a topological space and $x \in X$, we denote by $\mathcal{V}$ the set of open neighborhoods of x, which is filtering for the opposite order to inclusion i.e, for all $U, V \in \mathcal{V}$ we have*

$$U \leq V \Longleftrightarrow V \subseteq U.$$

Remark 1.1. *We can identify $\mathcal{F}_x$ as the quotient of the set of pairs (U, s), where $U \in \mathcal{V}$ and where s is a section of $\mathcal{F}$ on U, by the relation of equivalence defined as follows :*

$(U, s) \sim (V, t)$ if and only if there exists an open neighborhood W of x in $U \cap V$ such that $s_{|W} = t_{|W}$.

Moreover, we can see $\mathcal{F}_x$ as the set of sections of $\mathcal{F}$ defined in the neighborhood of x. Two sections belonging to $\mathcal{F}_x$ being considered as equal if they coincide in some neighborhood of x.

Example 1.2. *Let $\mathcal{F}(U) = \big\{$ Continuous functions $U \longrightarrow \mathbb{R}\big\}$. Then $\mathcal{F}_x$ the set of germs of continuous functions at x.*

Proposition 1.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves, then ψ induces for every point $x \in X$ a morphism $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ between the stalks, where ψ_x is defined by $\psi_x(s_x) = \big(\psi(U)(s)\big)_x$ for any open subset U of X, $s \in \mathcal{F}(U)$, and $x \in U$.*

Proof. *If $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are such that $s_x = t_x$, then there exists an open neighborhood W of x such that $s_{|W} = t_{|W}$. So $\psi(U)(s)_{|W} = \psi(W)(s_{|W})$ and $\psi(V)(t)_{|W} = \psi(V)(t_{|W})$. Hence $\big(\psi(U)(s)\big)_x = (\psi(V)(t))_x$.*

Note that if $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ and $\phi : \mathcal{G} \longrightarrow \mathcal{Z}$ are two morphisms of sheaves we have $(\psi \circ \phi)_x = \psi_x \circ \phi_x$ and $(id_{\mathcal{F}})_x = id_{\mathcal{F}_x}$. Moreover, $\psi \longrightarrow \psi_x$ define a functor from the category of sheaves over X to the category $\mathcal{C}$.

Definition 1.5. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves*

- *i) We say that ψ is injective if for any open subset U of X, $\psi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is injective.*
- *ii) We say that ψ is surjective if for all $x \in X$, $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is surjective.*

2. Sheaves

Definition 2.1. *We say that a presheaf $\mathcal{F}$ is a sheaf if we have the following properties:*

- *i) (Uniqueness) Let U be an open subset of X, $s \in \mathcal{F}(U)$, $\big\{U_i\big\}_{i \in I}$ a covering of U by open subsets U_i. If $s_{|U_i} = 0$ for every $i \in I$, then $s = 0$.*

ii) (Gluing axiom) If $U = \bigcup_{i\in I} U_i$, and if $s_i \in \mathcal{F}(U_i)$ is a collection of sections matching on the overlaps; that is, they satisfy

$$s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$$

for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$ so that $s_{|U_i} = s_i$, for all $i \in I$

Remarks 2.1. *1) When $\mathcal{F}$ is a presheaf of groups or of an algebraic structure that is in particular a group, we can replace i) by: for all $s, t \in \mathcal{F}(U)$ such that for $i \in I$, $s_{|U_i} = t_{|U_i}$ then $s = t$.*
2) The section s in ii) is unique by condition i).

Examples 2.1. *1) Let X be a topological space, $U \longmapsto \mathcal{C}^0(U, \mathbb{R})$ is a sheaf of $\mathbb{R}$-algebras over X.*
2) In example 1.1, if moreover, $\mathcal{F}$ is a sheaf then $\mathcal{F}_{|U}$ is still a sheaf.

2.1. Morphisms of sheaves.

Definition 2.2. *A morphism of sheaves is just a morphism of the underlying presheaves.*

Remarks 2.2. *1) The sheaves of X form a full subcategory $\mathcal{S}h_X$ of category of the presheaves on X.*
2) The notions injective, surjective and isomorphism for sheaves are defined in the same way as for presheaves.

Lemma 2.1. *Let X be a topological space and let U be an open subset of X.*
1) Let $\mathcal{F}$ be a sheaf on X and let $s, t \in \mathcal{F}(U)$ be two sections such that $s_x = t_x$ for every $x \in U$. Then $s = t$.
2) Let $\mathcal{F}$, $\mathcal{G}$ be presheaves on X and let $\psi, \phi : \mathcal{F} \longrightarrow \mathcal{G}$ be morphisms of presheaves on X such that $\mathcal{F}_x = \mathcal{G}_x$ for every $x \in X$. If $\mathcal{G}$ is a sheaf, then $\mathcal{F} = \mathcal{G}$.

Proof. *1) Let $x \in U$, since $s_x = t_x$, there exists an open subset W_x of U containing x such that $s_{|W_x} = t_{|W_x}$. Since $(W_x)_x$ is an open covering of U, according to condition i) in definition 2.1, it comes that $s = t$.*
2) Let W be an open subset of X and let $s \in \mathcal{F}(W)$. We need to prove that s has the same image under the maps $\psi(W)$ and $\phi(W)$, let $t = \psi(U)(s)$ and $l = \phi(U)(s)$. For all $x \in W$, we have $t_x = \psi_x(s_x) = \phi_x(s_x) = l_x$. Since $\mathcal{G}$ is a sheaf, so by 1) we get that $t = l$.

Proposition 2.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then ψ is injective if and only if $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is injective for every $x \in X$.*

Proof. *Suppose ψ is injective. Let $x \in X$ and $s_x \in \mathcal{F}_x$ such that $\psi_x(s_x) = 0$, where $s \in \mathcal{F}(U)$ and U is an open neighborhood of x, so $\big(\psi(U)(s)\big)_x = 0$. Then, there exists an open neighborhood W of x such that $\psi(U)(s)_{|W} = 0$ or that $\psi(W)(s_{|W}) = 0$. From the injectivity of ψ it comes that $s_{|W}$, thus $s_x = 0$. Conversely, suppose that for all $x \in X$, ψ_x is injective, we fix an open subset V of X and $s \in \mathcal{F}(V)$ such that $\psi(V)(s) = 0$, locally we have, for all $x \in V$, $\psi_x(s_x) = \big(\psi(U))(s)\big)_x = 0$, it comes from local injectivity, that for all $x \in V$, $s_x = 0$. Hence $s = 0$.*

Remark 2.1. *Proposition 2.1 gives a local characterization of the injectivity.*

Theorem 2.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. The following assertions are equivalent :*

1) ψ is an isomorphism.
2) For every $x \in X$, $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is an isomorphism.
3) ψ is both injective and surjective.

Proof. 1) $\Rightarrow$ 2) *Let ϕ be the inverse morphism of ψ. Plainly, for every $x \in X$, we have $\phi_x \circ \psi_x = id_{\mathcal{F}_x}$ and $\psi_x \circ \phi_x = id_{\mathcal{G}_x}$. So ψ_x is an isomorphism.*
2) $\Rightarrow$ 3) *Immediate, according to proposition 2.1 and definition 1.5, 2)*
3) $\Rightarrow$ 1) *We will construct the inverse ϕ of ψ. Let W be an open subset of X and $t \in \mathcal{G}(W)$, for every $x \in W$, there exists U_x an open neighborhood of x and $s^x \in \mathcal{F}(U_x)$ such that $t_x = \psi_x(s^x_x) = \big(\psi(U_x)(s^x)\big)_x$. Hence there exists $V_x \subseteq U_x \cap W$ neighborhood of x such that $t_{|V_x} = \big(\psi(V_x)(s^x_{|V_x})\big)_{|V_x}$. If $y \in W$, then $\psi(V_x \cap V_y)(s^x_{|V_x \cap V_y}) = \psi(V_x \cap V_y)(s^y_{|V_x \cap V_y})$, so $s^x_{|V_x \cap V_y} = s^y_{|V_x \cap V_y}$, as the family $(V_x)_{x \in U}$ forms a covering of U, then $(s^x)_x$ rises to a section s of $\mathcal{F}$ on U, and we have $\psi(U)(s) = t$, the uniqueness of s follows from the injectivity of ψ. We set $\phi(U)(t) = s$, then ϕ is the inverse of ψ.*

3. Sheafification

In this section, we answer the following question : How to build a sheaf from a presheaves?

Definition 3.1. *Let $\mathcal{F}$ be a presheaf on a topological space X. We call associated sheaf with $\mathcal{F}$ any sheaf $\mathcal{F}^\dagger$ equipped with a morphism of presheaves $\beta : \mathcal{F} \longrightarrow \mathcal{F}^\dagger$ satisfying the following universal property:*
For any morphism of presheaves $\psi : \mathcal{F} \longrightarrow \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there exists a unique morphism of sheaves $\overline{\psi} : \mathcal{F}^\dagger \longrightarrow \mathcal{G}$ such that the following diagram is commutative :

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\psi"] \arrow[d, "\beta"'] & \mathcal{G} \\ \mathcal{F}^\dagger \arrow[ur, "\overline{\psi}"'] & \end{tikzcd}$$

Remark 3.1. *The uniqueness of $\mathcal{F}^\dagger$ when it exists is an immediate consequence of the universal property.*

Proposition 3.1. *Let $\mathcal{F}$ be a presheaf on a topological space X. Then the sheaf $\mathcal{F}^\dagger$ associated with $\mathcal{F}$ exists and is a unique up to isomorphism. Moreover, using the above notation, all $x \in X$, the induced morphism $\beta : \mathcal{F}_x \longrightarrow \mathcal{F}^\dagger_x$ is an isomorphism.*

Proof. *Let $\mathcal{F}$ be a presheaf on X. Consider $Z := \coprod_{x \in X} \mathcal{F}_x$ (disjoint union) and consider the map $\pi : Z \longrightarrow X$ defined by : for all s_x, $\pi(s_x) = x$. For any open V of X and $s \in \mathcal{F}(V)$, let π_s be the map $\pi_s : V \longrightarrow X$ defined by $\pi_s(x) = s_x$. Note that $\pi\big(\pi_s(x)\big) = x$ i.e $\pi \circ \pi_s = id_U$ (π_s is a section and π is a retraction). We now endow*

Z with the topology which makes all maps $\pi_s : V \longrightarrow Z$, V open subset of X and $s \in \mathcal{F}(V)$, continuous.
For any open subset V of X, we define $\mathcal{F}^\dagger(V) := \{g : V \longrightarrow Z / g$ continuous and $\pi \circ g = id_V\}$ it is the set of sections of Z on V.

* *For every $W \subseteq V$, the restriction $\mathcal{F}^\dagger(V) \longrightarrow \mathcal{F}^\dagger(W)$ is the usual restriction, i.e $g \longrightarrow g_{|W}$. In particular $\mathcal{F}^\dagger$ is a presheaf.*
* *Condition i) in definition 2.1 is immediate.*
* *If $(W_j)_j$ is a covering of V and $g_j \in \mathcal{F}^\dagger(W_j)$ are such that for all i, j, $g_{i|W_i \cap W_j} = g_{j|W_i \cap W_j}$, then as the g_j are continuous, and coincide on the intersections, there exists $g : V \longrightarrow X$ which is continuous such that for all j, $g_{|W_j} = g_j$. Moreover g is a section in fact : for all $x \in V$, there is some j such that $x \in W_j$, $\pi \circ g(x) = \pi(g(x)) = \pi(g_j(x)) = x$.*
  $\mathcal{F}^\dagger$is a sheaf.
* *Definition of $\beta : \mathcal{F} \longrightarrow \mathcal{F}^\dagger$: For any open subset V of X and $s \in \mathcal{F}(V)$, we define $\beta(V)(S) := \pi_s \in \mathcal{F}^\dagger(V)$.*
* *Compatibility with restrictions : let $W \subseteq V$ two open subsets of X, $s \in \mathcal{F}(V)$ and $x \in W$, we have $\beta(V)(s)_{|W}(x) = \pi_s(x) = s_x = (s_{|W})(x) = \pi_{s|W}(x)$. So $\beta(V)(s)_{|W} = \beta(W)(s_{|W})$.*
* *Let $\mathcal{G}$ be a sheaf, and $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves. We cut a section g of $\mathcal{F}^\dagger(V)$ into small sections (sections of $\mathcal{F}$) on a covering W_j of V, then by sending them to the $\mathcal{G}(W_j)$, then we stick back into $\mathcal{G}$. Sections of $\mathcal{F}^\dagger$ are obtained by gluing sections of $\mathcal{F}$, so $\mathcal{F}_x = \mathcal{F}^\dagger_x$.*

Remark 3.2. *If $\mathcal{F}$ is a sheaf, it follows from the universal property that $\mathcal{F} \simeq \mathcal{F}^\dagger$.*

Example 3.1. *Let A be a group (or a ring, an algebra,...), then*

$$U \longmapsto \begin{cases} A & \text{if} \qquad U \neq \emptyset \\ \{0\} & \text{otherwise} \end{cases}$$

is a presheaf and the associated sheaf is called the constant sheaf associated to A. We denoted by $\underline{A}$. For any $x \in X$, we have $\underline{A}_x = A$.

4. Subsheaves and Quotient sheaves

Throughout, we fix a category of objects that have an algebraic structure which are in particular groups, say e.g., $\mathcal{C} = \mathcal{G}p$ or R-$\mathcal{M}od$.

4.1. **Subsheaves.**

Definition 4.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X, we say that $\mathcal{F}$ is a subsheaf of $\mathcal{G}$, if for any open subset U of X, $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ and such that we have compatibility with the restrictions induced from $\mathcal{F}$ and $\mathcal{G}$, i.e., For every open subsets $U \subseteq V$ of X, the following diagram is commutative:*

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, hook] \arrow[d, "res_{V,U}"'] & \mathcal{G}(V) \arrow[d, "res_{V,U}"] \\ \mathcal{F}(U) \arrow[r, hook] & \mathcal{G}(U) \end{tikzcd}$$

Remark 4.1. *$\mathcal{F}$ is a subsheaf of $\mathcal{G}$ if, the canonical injection $\imath : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves.*

Definition 4.2. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ a morphism of presheaves on X. We define the presheaf $ker(\psi)$ by the formula :*

$$U \longrightarrow ker(\psi(U))$$

for any open subset U of X. $ker(\psi)$ is said to be the kernel of ψ, it's a subpresheaf of $\mathcal{F}$. and ψ is injective if and only if its kernel is the trivial presheaf.

Using the notation of Definition 4.2, one can easily see that ψ is injective if and only if its kernel is the trivial presheaf.

Lemma 4.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then the presheaf $ker(\psi)$ is a sheaf.*

Proof. *Let U be an open of X , $(U_j)_j$ be a covering of U and $s_j \in ker(\psi(U_j))$ such that for i, j, $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$. Since $s_j \in \mathcal{F}(U_j)$, then $(s_j)_j$ rises to a section s of $\mathcal{F}$ over U, but for every $x \in U$, there exists i such that $x \in U_j$, and we have $\big(\psi(U)\big)(s)_x = \big(\psi(U_j)\big)(s_j)_x = 0$. So $\psi(U)(s) = 0$. Hence $s \in ker(\psi(U))$. On the other hand, if $s \in ker(\psi(U))$ such that for every j, $s_{|U_j} = 0$, then $s = 0$ (because $s \in \mathcal{F}(U)$ and $\mathcal{F}$ is a sheaf).*

Definition 4.3. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves on X. We define the $im(\psi)$ presheaf by the formula :*

$$U \longmapsto im\big(\psi(U)\big)$$

for any open set U of X. One can easily see that $im(\psi)$ is indeed a subpresheaf of $\mathcal{G}$. We say that $im(\psi)$ is the image presheaf of ψ.

Remark 4.2. *Note that the presheaf $im(\psi)$ is not in general a sheaf. In the same way we define the presheaf $U \longmapsto coker - pr(im(\psi))$ which too is not in general a sheaf. This justifies the following definition.*

Definition 4.4. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaf. The sheaf associated with the image presheaf $im - pr(\psi)$ called the image sheaf of ψ is denoted $im(\psi)$. In the same way we define the cokernel sheaf and that we denote by $coker(\psi)$*
Note that in general $\big(im(\psi)\big)(U) \neq im\big(\psi(U)\big)$. The first term is section of the sheaf $im(\psi)$ on the open set U, while the second is the image of the morphism $\psi(U)$. More precisely, we have :

Theorem 4.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then, the following assertions hold :*

1) *For any open subset U of X, and $s \in \mathcal{G}(U)$. $s \in (im(\psi)(U))$ if and only if there exists an open covering (U_j) of U and $t_j \in \mathcal{F}(U_j)$ such that, for any j, $s_{|U_j} = \psi(U_j)(t_j)$.*
2) *ψ is surjective if and only if, for any open subset U of X and $s \in \mathcal{G}(U)$, there exists an open covering $(U_j)_j$ of U and $t_j \in \mathcal{F}(U_j)$ such that, for any j, $s_{|U_j} = \psi(U_j)(t_j)$.*

3) ψ is surjective if and only if $\mathcal{G} = im(\psi)$.

Proof. *1) $im(\psi)$ is a the sheaf associated with presheaf $U \longmapsto im\big(\psi(U)\big)$, hence the result.*

2) If ψ is surjective, let U an open subset of X and $s \in \mathcal{G}(U)$, for all $x \in U$, by theorem 2.1, the map ψ_x is surjective. So there exists $t_x \in \mathcal{F}_x$ such that $\psi_x(t_x) = s_x$. Therefore, there there exists an open neighborhood $U_x \subseteq U$, and $t^x \in U_x$ such that $s_{|U_x} = \psi(U_x)(t^x)$. The covering $(U_x)_{x\in U}$ answers the question. Conversely, let $x \in X$ and $s \in \mathcal{G}(U)$. Let $(U_j)_j$ be covering of U and $t_j \in \mathcal{F}(U_j)$ such that $s_{|U_j} = \psi(U_j)(t_j)$ for all j. Since $\mathcal{F}$ is a sheaf then there is $t \in \mathcal{F}(U)$ such that $t_{|U_j} = t_j$ for all j. In particular, for every j such that $x \in U_j$, $s_x = (s_{|U_j})_x = \big(\psi(U_j)(t_j)\big)_x = \psi_x(t_x)$. Hence ψ is surjective.

3) Immediate from 1) and 2).

4.2. **Quotients sheaves.** Assume that $\mathcal{F}$ is a subsheaf of the sheaf $\mathcal{G}$. Then we can define a presheaf whose sections over U are the quotient $\mathcal{G}(U)/\mathcal{F}(U)$. The restriction maps of $\mathcal{F}$ and $\mathcal{G}$ are compatible the inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ and hence pass to the quotient $\mathcal{G}(U)/\mathcal{F}(U)$. This presheaf, i.e., $U \longmapsto \mathcal{G}(U)/\mathcal{F}(U)$, is called quotient presheaf of $\mathcal{G}$ by $\mathcal{F}$.

Definition 4.5. *The quotient sheaf $\mathcal{G}/\mathcal{F}$ is the sheafification of the quotient presheaf of $\mathcal{G}$ by $\mathcal{F}$.*

Proposition 4.1. *Let $\mathcal{F}$ be a subsheaf of $\mathcal{G}$, $x \in X$. Then $(\mathcal{G}/\mathcal{F})_x = \mathcal{G}_x/\mathcal{F}_x$.*

Proof. *$\mathcal{G}/\mathcal{F}$ is the sheaf associated with the presheaf $U \longmapsto \mathcal{G}(U)/\mathcal{F}(U)$ whose stalks at x is clearly isomorphic to $\mathcal{G}_x/\mathcal{F}_x$.*

5. Continuous maps and sheaves

5.1. Pushforward.

Definition 5.1. *Let $f : Y \longrightarrow X$ be a continuous map between topological spaces. Let $\mathcal{F}$ be a presheaf on X. We define the pushforward of $\mathcal{F}$ by the formula :*

$$f_*\mathcal{F}(V) = \mathcal{F}\big(f^{-1}(V)\big)$$

for any open $V \subseteq Y$.
Given opens $W \subseteq V$ of Y open the restriction map is given by the commutativity of the diagram

$$\begin{tikzcd}
f_*\mathcal{F}(V) \arrow[r, equal] \arrow[d] & \mathcal{F}\big(f^{-1}(V)\big) \arrow[d, "res_{f^{-1}(V),f^{-1}(W)}"] \\
f_*\mathcal{F}(W) \arrow[r, equal] & \mathcal{F}\big(f^{-1}(W)\big)
\end{tikzcd}$$

It is clear that this defines a presheaf on Y.

Remark 5.1. *The construction is clearly functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor*

$$\begin{array}{cccc} f_*: & \mathcal{P}reSh_X & \longrightarrow & \mathcal{P}reSh_Y \\ & \mathcal{F} & \longmapsto & f_*\mathcal{F} \end{array}$$

Proposition 5.1. *Let $f : X \longrightarrow Y$ be a continuous map and $\mathcal{F}$ be a sheaf on X. Then $f_*\mathcal{F}$ is a sheaf on Y.*

Proof. *This immediately follows from the fact that if $(W_j)_j$ is an open covering of some open subset W of Y then, $\bigcup_j f^{-1}(W_j)$ is an open covering of the open $f^{-1}(W)$. Consequently, we obtain a functor*

$$f_* : \mathcal{S}h_X \longrightarrow \mathcal{S}h_Y$$

This is compatible with composition in the following strong sense :

Lemma 5.1. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous maps of topological spaces. Then, the functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal.*

Proof. *Immediate.*

5.2. **Pullback.** We saw in example 1.1 that if $\mathcal{F}$ is a sheaf on X, then for any open subset U of X $\mathcal{F}_{|U}$ is a sheaf on U. Now if we take an arbitrary subset Z of X. the restriction of $\mathcal{F}$ on Z is not necessarily a sheaf because an open set W of Z is not necessarily an open set of X. Next definition gives the meaning of $\mathcal{F}_{|Z}$, when Z is a closed subset of X. This will be generalized in Definition 5.3 to give the meaning of the pullback presheaf defined by a continuous map. For this purpose, note that if $f : X \longrightarrow Y$ is a continuous map between topological spaces and V is an open of Y, then the family $(U)_{f(U)\subseteq V}$ consisting of all open subsets U of X satisfying $f(U) \subseteq V$, is an inductive system for the inverse of the inclusion relation.

Definition 5.2. *If $\imath : Z \longrightarrow X$ is the inclusion of a closed subset Z of X, and V is an open subset of Z. We define the restriction $\mathcal{F}_{|Z}$ as the sheafification of the following presheaf*

$$V \longmapsto \varinjlim_{V \subseteq U} \mathcal{F}(U)$$

Definition 5.3. *Let $f : X \longrightarrow Y$ be a continuous map between topological spaces and $\mathcal{G}$ be a presheaf on Y. We define the pullback presheaf of $\mathcal{G}$ by the formula :*

$$f_p\mathcal{G}(U) = \varinjlim_{f(U)\subseteq V} \mathcal{G}(V).$$

Remark 5.2. *In the language of categories. The pullback presheaf $f_p\mathcal{G}$ of $\mathcal{G}$ is defined as the left adjoint of the pushforward f_* on presheaves. In other words, $f_p\mathcal{G}$ will be a presheaf on X such that*

$$Mor_{\mathcal{P}reSh_X}(f_p\mathcal{G}, \mathcal{F}) = Mor_{\mathcal{P}reSh_Y}(G, f_*\mathcal{F})$$

Proposition 5.2. *Let $f : X \longrightarrow Y$ be a continuous map between topological spaces, x be a point of X and $\mathcal{G}$ be a presheaf on $\mathcal{Y}$. Then, up to an isomorphism, we have $(f_p\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof.

$$\begin{aligned}(f_p\mathcal{G})_x &= \varinjlim_{x\in U} f_p\mathcal{G}(U)\\ &= \varinjlim_{x\in U}\varinjlim_{f(U)\subseteq V}\mathcal{G}(V)\\ &= \varinjlim_{f(x)\in V}\mathcal{G}(V)\\ &= \mathcal{G}_{f(x)}\end{aligned}$$

Definition 5.4. *Let $f : X \longrightarrow Y$ be a continuous map between topological spaces and $\mathcal{G}$ be a sheaf on Y. The pullback sheaf $f^{-1}\mathcal{G}$ is defined by the formula :*

$$f^{-1}\mathcal{G} = (f_p\mathcal{G})^{\dagger}$$

$f^{-1}\mathcal{G}$ is also called the inverse image along the map f.

Remark 5.3. *f^{-1} defines a functor :*

$$\begin{array}{cccc} f^{-1}: & \mathcal{S}h_Y & \longrightarrow & \mathcal{S}h_X\\ & \mathcal{G} & \longmapsto & f^{-1}\mathcal{G}\end{array}$$

The pullback f^{-1} is a left adjoint of pushforward on sheaves.

$$Mor_{\mathcal{S}h_X}(f^{-1}\mathcal{G},\mathcal{F}) = Mor_{\mathcal{S}h_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Example 5.1. *Let $\mathcal{F}$ be a sheaf on X and $x \in X$. Let $\imath : \{x\} \longrightarrow X$ be the inclusion map, then $\imath^{-1}\mathcal{F} = \mathcal{F}_x$*

Lemma 5.2. *Let $f : X \to Y$ be a continuous map between topological spaces, $x \in X$ and $\mathcal{G}$ be a sheaf on Y, then the stalks $(f^{-1}\mathcal{G})_x$ and $\mathcal{G}_{f(x)}$ are equals.*

Proof. *This a combination of proposition 3.1 and proposition 5.2.*

Lemma 5.3. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous maps of topological spaces. The functors $(g\circ f)^{-1}$ and $f^{-1}\circ g^{-1}$ are canonically isomorphic. Similarly, $(g\circ f)_p = f_p \circ g_p$, for presheaves.*

Proof. *This follows from the fact that adjoint functors are unique up to unique isomorphism, and Lemma 5.1.*

6. Exact sequences of sheaves

Definition 6.1. *A sequence of presheaves with presheaves morphisms*

$$\cdots \longrightarrow \mathcal{F}^{j-1} \xrightarrow{\psi^{j-1}} \mathcal{F}^{j} \xrightarrow{\psi^{j}} \mathcal{F}^{j+1} \xrightarrow{\psi^{j+1}} \cdots$$

is said to be exact if for all i, $Im(\psi^{j-1}) = ker(\psi^j)$. In particular the following exact sequence is call a short exact sequence when it is exact :

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

Remark 6.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then by,* *i) ψ is injective if and only if*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\psi} \mathcal{G}$$

is an exact sequence.

ii) ψ is surjective if and only if

$$\mathcal{F} \xrightarrow{\psi} \mathcal{G} \longrightarrow 0$$

is an exact sequence.

Example 6.1. *Let $X = \mathbb{C}$, and $\mathcal{O}_X$ the sheaf of holomorphic functions and consider the map $d : \mathcal{O}_X \longrightarrow \mathcal{O}_X$, sending $f(z)$ to $f^{'}(z)$. There is an exact sequence*

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X \xrightarrow{d} \mathcal{O}_X \longrightarrow 0$$

Indeed,

- *A function whose derivative vanishes identically is locally constant, so $ker(d)$ is the constant sheaf $\mathbb{C}_X$.*
- *In small open disks any holomorphic function is a derivative.*

Lemma 6.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves on X. Then for any $x \in X$, we have $(ker\psi)_x = ker(\psi_x)$ and $(im\psi)_x = im(\psi_x)$.*

Proof. *Let $s_x \in \big(ker(\psi)\big)_x$, and let U an open neighborhood of x such that $s \in \big(ker(\psi)\big)(U) = ker\big(\psi(U)\big)$, so $\psi(U)(s) = 0$, hence $\psi_x(s_x) = \big(\psi(U)(s)\big)_x = 0$, so $s_x \in ker(\psi_x)$. Conversely, if $\psi_x(s_x) = 0$, then $\big(\psi(U)(s)\big)_x = 0$ (U is an open neighborhood of x and $s \in \mathcal{F}(U)$), then there exists an open neighborhood $W \subseteq U$ of x such that $\psi(U)(s)_{|W} = 0$, it comes while $\psi(W)(s_{|W}) = 0$ and therefore $s_{|W} \in ker\big(\psi(W)\big)$ whence $s_x = (s_{|V})_x \in \big(ker\big(\psi\big)\big)_x$. One can proceed similarly for the image.*

Theorem 6.1. *A sequence of sheaves with sheaves morphisms*

$$\cdots \longrightarrow \mathcal{F}^{j-1} \xrightarrow{\psi^{j-1}} \mathcal{F}^{j} \xrightarrow{\psi^{j}} \mathcal{F}^{j+1} \xrightarrow{\psi^{j+1}} \cdots$$

is an exact sequence if and only if for any $x \in X$

$$\cdots \longrightarrow \mathcal{F}_x^{j-1} \xrightarrow{\psi_x^{j-1}} \mathcal{F}_x^{j} \xrightarrow{\psi_x^{j}} \mathcal{F}_x^{j+1} \xrightarrow{\psi_x^{j+1}} \cdots$$

is an exact sequence.

Proof.

$$\cdots \longrightarrow \mathcal{F}^{j-1} \xrightarrow{\psi^{j-1}} \mathcal{F}^{j} \xrightarrow{\psi^{j}} \mathcal{F}^{j+1} \xrightarrow{\psi^{j+1}} \cdots$$

is exact sequence if and only if, for any j, $im(\psi^{j-1}) = ker(\psi^j)$ if and only if, for any $x \in X$ and for any j, $im(\psi_x^{j-1}) = ker(\psi_x^j)$ if and only if,

$$\cdots \longrightarrow \mathcal{F}_x^{j-1} \xrightarrow{\psi_x^{j-1}} \mathcal{F}_x^{j} \xrightarrow{\psi_x^{j}} \mathcal{F}_x^{j+1} \xrightarrow{\psi_x^{j+1}} \cdots$$

is exact sequence.

Proposition 6.1. *Let $\mathcal{F}$ be a subsheaf of $\mathcal{G}$ on X. Then*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0$$

is exact sequence.

Proof. *By proposition 4.1, for any $x \in X$,*

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{G}_x/\mathcal{F}_x = (\mathcal{G}/\mathcal{F})_x \longrightarrow 0$$

is exact sequence. Hence the result.

Remark 6.2. *If*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence over X, then $\mathcal{F}$ identified with a sub-sheaf of $\mathcal{G}$ and $\mathcal{G}/\mathcal{F} \simeq \mathcal{H}$.

Corollary 6.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then*

1) $im(\psi) \simeq \mathcal{F}/ker(\psi)$.
2) $coker(\psi) \simeq \mathcal{G}/im(\psi)$.

Proof. *1) It is easy to check that for all $x \in X$, we have*

$$0 \longrightarrow \big(ker(\psi)\big)_x \longrightarrow \mathcal{F}_x \longrightarrow im(\psi)_x \longrightarrow 0$$

It follows by theorem 6.1, that

$$0 \longrightarrow ker(\psi) \longrightarrow \mathcal{F} \longrightarrow im(\psi) \longrightarrow 0$$

is an exact sequence. Also by remark 6.2 we have $im(\psi) \simeq \mathcal{F}/ker(\psi)$
2) Similar to 1).

7. Glueing sheaves

In this section, we fix a topological space X, and we consider an open covering $\{U_i\}_{i\in I}$ of X with a sheaf $\mathcal{F}_i$ on each subset U_i. Our goal is to "glue" the $\mathcal{F}_i$ together, that is we search for a global sheaf $\mathcal{F}$ such that $\mathcal{F}_{|U_i} = \mathcal{F}_i$ for all $i \in I$.

Notation. *i) For $i, j \in I$, we denote by U_{ij} the intersection $U_i \cap U_j$.*
ii) For $i, j, k \in I$, we denote by U_{ijk} the intersection $U_i \cap U_j \cap U_k$.

Definition 7.1. *A Gluing Datum consists of a family of sheaves $\mathcal{F}_i$ over U_i and a family of morphisms $\delta_{ij} : \mathcal{F}_{i|U_{ij}} \longrightarrow \mathcal{F}_{j|U_{ij}}$ such that*

i) $\delta_{ii} = id_{\mathcal{F}_i}$.
ii) $\delta_{ji} = \delta_{ij}^{-1}$.
iii) $\delta_{ik} = \delta_{jk} \circ \delta_{ij}$ on U_{ijk}.

A morphism of gluing datum $(\mathcal{F}_i, \delta_{ij}) \longrightarrow (\mathcal{G}_i, \eta_{ij})$ *is a family of morphism of sheaves* $\psi_i : \mathcal{F}_i \longrightarrow \mathcal{G}_i$ *such that the following diagram*

$$\begin{tikzcd} \mathcal{F}_i \arrow[r, "\psi_i"] \arrow[d, "\delta_{ij}"'] & \mathcal{G}_i \arrow[d, "\delta_{ij}"] \\ \mathcal{F}_j \arrow[r, "\psi_i"] & \mathcal{G}_j \end{tikzcd}$$

is commutative.

Theorem 7.1. *(Gluing sheaves) There exists a sheaf* $\mathcal{F}$ *on* X*, unique up to ismorphism such that there are isomorphisms* $\theta_i : \mathcal{F}_{U_i} \longrightarrow \mathcal{F}_i$ *such that there are satisfying*

$$\theta_j = \delta_{ij} \circ \theta_i.$$

Proof. *Let* W *be an open subset of* X*. We write* $W_i = U_i \cap W$*, and* $W_{ij} = U_{ij} \cap W$*. We are going to define the sections of* $\mathcal{F}$ *over* W *by gluing sections of the* $\mathcal{F}_i'$*s over* W_i'*s along the* W_{ij}'*s using the isomorphisms* δ_{ij}*. We define*

$$\mathcal{F}(W) := \big\{(s_i)_{i\in I} | \delta_{ji}(s_{i|W_{ij}}) = \delta_{j|W_{ij}}(s_{j|W_{ij}})\big\} \subseteq \prod_{i\in I} \mathcal{F}_i(W_i). \tag{7.1}$$

The δ_{ij}*'s are morphisms of sheaves and therefore are compatible with all restrictions maps (see definition 1.3). So if* $V \subseteq W$ *is another open subset we have*

$$\delta_{ij}(s_{i|W_{ij}}) = s_{j|W_{ij}}.$$

Because of this, the defining condition (7.1) is compatible with componentwise restrictions, and they can therefore be used as the restriction maps in $\mathcal{F}$*. So We have defined a presheaf on* X*.*

* *The first step: is to establish the isomorphisms* $\theta_i : \mathcal{F}_{|U_i} \longrightarrow \mathcal{F}_i$*. To avoid getting confused by the names of the indices, we shall work with a fixed index* $j \in I$*. Suppose* $W \subseteq U_j$ *is an open set. We have* $W = W_j$*, and projecting from the product* $\prod_{i\in I} \mathcal{F}_i(W_i)$ *onto the component*

  $$\mathcal{F}_j(W) = \mathcal{F}_j(W_j)$$

  gives us a map $\theta : \mathcal{F}_{j|W_j} \longrightarrow \mathcal{F}_j$*. Moreover,* $\theta\big((s_i)_{i\in I}\big) = s_j$*. The situation is summarized in the following commutative diagram*

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, hook] \arrow[dr, "\theta_j"'] & \prod_{i\in I} \mathcal{F}_i(W_i) \arrow[d, "\pi_j"] \\ & \mathcal{F}_j(W) \end{tikzcd}$$

Now, we want to show that θ_j*'s give the desired isomorphisms. We note that on the restrictions* $W_{jj'}$*, the requirement in the proposition, that*

$$\theta_{j'} = \eta_{j'j} \circ \theta_j$$

is fulfilled. This follows directly from the (7.1) that

$$s_{j|W_{jj'}} = \delta_{jj'}(s_{j'|W_{jj'}}).$$

- $*$ *θ_j is surjective: Let α a section of $\mathcal{F}_j(W)$ over some $W \subseteq U_j$, and pose $s = (\delta_{ij}(\alpha_{|W_{ij}})_{i\in I}$. Then s satisfies (7.1) and is an element $\mathcal{F}(W)$. Indeed, by definition 7.1 iii) we obtain*

$$\delta_{ki}(\delta_{ij}\big(\alpha_{|W_{kij}})\big) = \delta_{kj}(\alpha_{|W_{kij}}).$$

  for each $i, k \in I$, and that is just the condition (7.1). As $\delta_{jj}(\alpha_{|W_{jj}}) = \alpha$ by the first gluing request, the element s projects to the section α of $\mathcal{F}_j$.
- $*$ *θ_j is injective : Since $s_j = 0$ if follows that $s_{i|W_{ij}} = \delta_{ij}(s_j) = 0$ for each $i \in I$. Now $\mathcal{F}_j$ is a sheaf, and the $\big\{V_{ij}\big\}_{i\in I}$ constitute an open covering of W_j, so we may conclude that $s = 0$ by definition 2.1 i).*
- $*$ *The final step: To show that $\mathcal{F}$ is a sheaf. Let $\big\{W_j\big\}_{j\in I}$ be an open covering of $W \subseteq U$, and $s_j \in \mathcal{F}(W_j)$ is a bunch of sections matching on the intersections $W_{jj'}$. Since $\mathcal{F}_{|U_i\cap W}$ is a sheaf patch together to give sections s_i in $\mathcal{F}_{U_i\cap W)}$ matching on the overlaps $U_{ij} \cap W$. This last condition means that $\delta ij(s_i) = s_j$. By definition $(s_i)i \in I$, then is a section in $\mathcal{F}(W)$ restricting to s_i. Hence the result.*

  The Gluing axiom (see definition 2.1) is easier : Let $s = (s_i)_{i\in I}$ in $\mathcal{F}(W)$, and a covering $\mathcal{L} = \big\{V_j\big\}_{j\in J}$ of W such that $s_{|V_j} = 0$ for all $j \in J$, then also $s_{|V_j\cap W_i} = 0$, and since $\big\{V_j \cap W_i\big\}_{j\in J}$ forms a covering of W_i, we must have $s_{|W_i} = 0$ as well, since $\mathcal{F}_{W_i} = \mathcal{F}_i$ is a sheaf. But from the (7.1) we thus see that $s = 0$.

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