Non-homogeneity of Lie and Jordan products in the matrix ring $M_4(k)$ under dihedral group D_{10} grading

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Abstract

In this note, we prove that the Lie and Jordan products of homogeneous elements in a graded ring remain homogeneous only if the grading group is abelian. To illustrate this, we construct an explicit counterexample using the matrix ring $M_4(k)$, graded by the non-abelian dihedral group D_{10}. In this case, the homogeneity of these products is not preserved.

Let R be an associative ring with center $Z(R)$, and let G be an abelian group with identity e. For $x, y \in R$, the symbol $[x, y]$ (resp. $x \circ y$) denotes the Lie product $xy - yx$ (resp. $xy + yx$ for Jordan product). A ring R is G-graded if there is a family $\{R_g, g \in G\}$ of additive subgroups R_g of $(R, +)$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for every $g, h \in G$. The additive subgroup R_g is called the homogeneous component of R, and we denote by $\mathcal{H}(R) = \bigcup_{g\in G} R_g$ the set of homogeneous elements of R. A nonzero element $x \in R_g$ is said to be homogeneous of degree g, and we write $\deg(x) = g$. Each element $x \in R$ has a unique decomposition $x = \sum_{g\in G} x_g$ with $x_g \in R_g$ for all $g \in G$, where the sum is finite. The terms x_g are called the homogeneous components of element x.

Proposition. If G is abelian, the Lie product and Jordan product of homogeneous elements are also homogeneous. More precisely, if $x \in R_g$ and $y \in R_h$, then:

$$[x, y] \in R_{gh} \quad \text{and} \quad x \circ y \in R_{gh}.$$

Proof. Straightforward. □

In the following example, when G is non-abelian, homogeneity is not preserved under Lie and Jordan products.

Example. Let $R = M_4(k)$ (the ring of 4×4 matrices with coefficients in the field k) and $G := D_{10} = \{a, b \mid a^5 = b^2 = e, ba = a^{-1}b\}$. We may define a G-grading on R by putting

$$R_e := \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix}, \quad R_a := \begin{pmatrix} 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{a^2} := \begin{pmatrix} 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_{a^3} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_b := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \end{pmatrix}, \quad R_{ab} := \begin{pmatrix} 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \end{pmatrix}$$

$$R_{a^4b} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & k & 0 \end{pmatrix}, \quad R_{a^4} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{a^2b} = R_{a^3b} = \{0\}.$$

It is straightforward to check that

$$[R_b, R_{a^4}] = R_b \circ R_{a^4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ k & 0 & 0 & 0 \end{pmatrix} \not\subset \mathcal{H}(R).$$